χ-négyzet próba

- H_0 hipotézis: az A_1, A_2 , ..., A_r teljes eseményrendszerre teljesül $P(A_1)=p_1$, $P(A_2)=p_2$, ..., $P(A_r)=p_r$
- A tesztstatisztika: $$\sum_{i=1}^{r} \frac{(\nu_i - np_i)^2}{np_i}$$

  ami aszimptotikusan r-1 szabadságfokú χ-négyzet eloszlású, ha igaz a nullhipotézis.
- Kritikus tartomány: ha a statisztika értéke nagyobb, mint az r-1 szabadságfokú χ-négyzet eloszlás 1- α kvantilise, elutasítjuk a nullhipotézist.

χ-négyzet próba (folytatás)

- Miért is ez a határeloszlás?

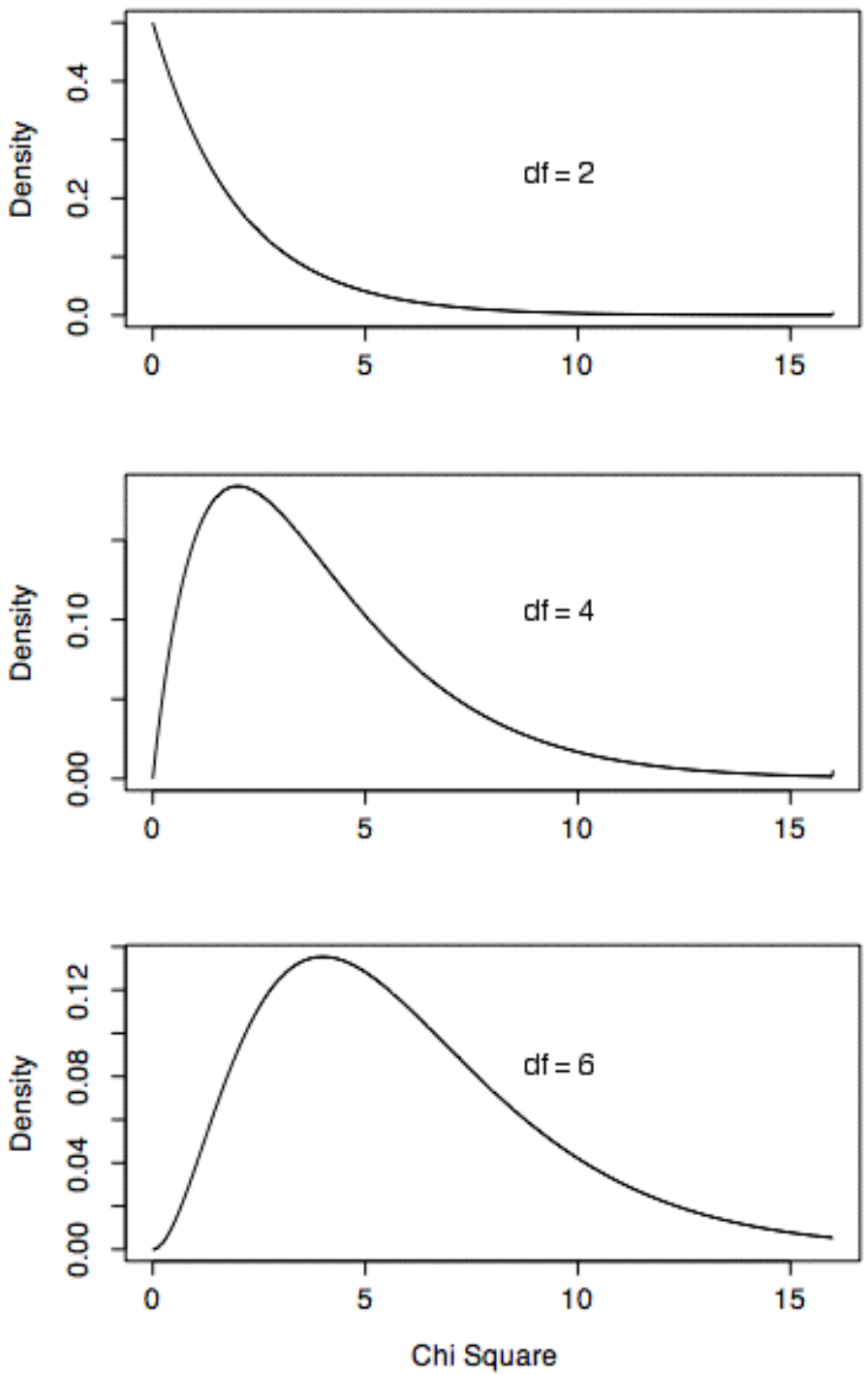
$r = 2,\ H_0 : P(A) = p,\ \nu : A$ gyakorisága n kísérletből

$$\chi^2 = \frac{(\nu - np)^2}{np} + \frac{((n-\nu) - n(1-p))^2}{n(1-p)} = \frac{(\nu - np)^2}{np} + \frac{(\nu - np)^2}{n(1-p)} = \frac{(\nu - np)^2}{np(1-p)}$$

$\xi_i = 1$, ha az $i.$ kísérletnél A bekövetkezik, 0 különben

$$\nu = \sum_{i=1}^{n} \xi_i,\ E\xi_i = p,\ D^2\xi_i = p(1-p),$$

$$\chi^2 = \left(\frac{\sum_{i=1}^{n} \xi_i - nE\xi_1}{\sqrt{n} D\xi_1} \right)^2 \xrightarrow[n \to \infty, \text{eloszlásban}]{} \chi_1^2$$


Density
0.0
0.2
0.4
df = 2
0
5
10
15
Density
0.00
0.10
df = 4
0
5
10
15
Density
0.00
0.04
0.08
0.12
df = 6
0
5
10
15
Chi Square

Példa (kockadobás)

- 36 kockadobás eredménye

Szám	Megfigyelt	np_i	$\frac{(\nu_i - np_i)^2}{np_i}$
1	8	6	0.667
2	5	6	0.167
3	9	6	1.500
4	2	6	2.667
5	7	6	0.167
6	5	6	0.167

$$n = 36,\ r = 6$$

$$\sum_{i=1}^{6} \frac{\left(\nu_i - np_i\right)^2}{np_i} \sim \chi_5^2$$

$$\sum_{i=1}^{6} \frac{\left(\nu_i - np_i\right)^2}{np_i} = 5.333$$

$$P\left(\chi_5^2 > 5.333\right) = 0.377 \Rightarrow$$

Nem tudjuk a szabályosság hipotézisét elutasítani!

Példa (számítógépek népszerűsége)

- 100 amerikai diák

Számí-tógép	Megfigyelt	np_i	$\frac{(\nu_i - np_i)^2}{np_i}$
IBM	47	33.333	5.604
Macintosh	36	33.333	0.213
Egyéb	17	33.333	8.003

$$n = 100,\ r = 3$$

$$\sum_{i=1}^{3} \frac{\left(\nu_i - np_i\right)^2}{np_i} \sim \chi_2^2$$

$$\sum_{i=1}^{3} \frac{\left(\nu_i - np_i\right)^2}{np_i} = 13.820$$

$$P\left(\chi_2^2 > 5.99\right) = 0.05 \Rightarrow$$

Elutasítjuk az egyforma kedveltség hipotézisét!

χ-négyzet próba illeszkedésvizsgálatra

- Illeszkedésvizsgálat:

$H_0 : \xi_1, ..., \xi_n \ \ F$ eloszlásfüggvényűek

- Visszavezetjük az előző esetre

$$A_i = \{\xi \in C_i\}, i = 1, 2, ..., r, \bigcup_i C_i = \mathbf{R}$$

Diszkrét esetben gyakran: $A_i = \{\xi = x_i\}, i = 1, 2, ..., r$

Példa

- Mi lehet egy vezető által okozott károk számának eloszlása?
- Poisson eloszlású-e?

Kár-szám	0	1	2	3	4	5	6	7	>7	Össze-sen
Veze-tők száma	129524	16267	1966	211	31	5	1	1	0	148006

Becsléses χ-négyzet próba

- H_0 hipotézis: az A_1, A_2 , ..., A_r teljes eseményrendszerre teljesül:

$$P(A_i) = p_i(\vartheta_1, \ldots, \vartheta_s), i = 1, 2, \ldots, r$$

$\vartheta_1, \ldots, \vartheta_s$ ismeretlen paraméterek.

A tesztstatisztika:

$$\chi^2 = \sum_{i=1}^{r} \frac{\left(\nu_i - n\hat{p}_i\right)^2}{n\hat{p}_i} \xrightarrow[n\to\infty]{} \chi^2_{r-s-1},$$

ahol

$$\hat{p}_i = p_i(\hat{\vartheta}_1, \ldots, \hat{\vartheta}_s).$$

Példa (folyt.)

Kár-szám	0	1	2	3	4	5	6	7	>7	Össze-sen
Veze-tők száma	129524	16267	1966	211	31	5	1	1	0	148006
np_i *Poisson*	128 433	18 218	1 292	61	2,2	0,06	0,001	3E-05	5E-07	
Np_i *Neg. bin.*	129 541	16 237	1 962	234	28	3,3	0,39	0,05	0,006	

$n = 148006,\ r = 5$

$A_i = \{\xi = i\}, i = 0,1,2,3$

$A_4 = \{\xi \geq 4\}$

Poisson eset:

$\hat{\lambda}$=0.709

$$\sum_{i=0}^{4} \frac{\left(\nu_i - n\hat{p}_i\right)^2}{n\hat{p}_i} \sim \chi^2_{5-1-1}$$

$$\sum_{i=0}^{4} \frac{\left(\nu_i - n\hat{p}_i\right)^2}{n\hat{p}_i} > 200$$

$$P\left(\chi_3^2 > 17.7\right) = 0.05\% \Rightarrow$$

Elutasítjuk Poisson eloszlás hipotézisét!

Az illeszkedésvizsgálat alkalmazása folytonos eloszlásokra

- A teljes eseményrendszer a számegyenes felosztása révén jön létre.
- Ügyeljünk arra, hogy minden intervallum közel azonos valószínűségű legyen.
- Ha paraméterbecslés szükséges, ML módszer alkalmazható.

χ-négyzet próba homogenitásvizsgálatra

- Homogenitásvizsgálat:

$H_0 : \xi_1, \ldots, \xi_n$ és $\eta_1, \ldots, \eta_m$ ugyanolyan eloszlásúak

- Hasonlóan járunk el, mint korábban

$$\bigcup_{i=1}^{r} C_i = \mathbf{R}$$

$$\nu_i = \left|\left\{ j : \xi_j \in C_i \right\}\right|, \mu_i = \left|\left\{ j : \eta_j \in C_i \right\}\right|, \; i = 1, 2, \ldots, r,$$

A tesztstatisztika:

$$\chi^2 = nm \sum_{i=1}^{r} \frac{\left(\nu_i / n - \mu_i / m \right)^2}{\nu_i + \mu_i} \xrightarrow[n,m \to \infty]{} \chi^2_{r-1}$$

Ki tanul jobban?

2009. január 5-ei vizsga

Jegy	Férfi	Nő	Összesen
1	47	4	51
2	11	1	12
3	11	2	13
4	9	2	11
5	8	2	10
Összesen	86	11	97
Átlag	2,1	2,7	2,1

$$C_1 = \{1;2\}, C_1 = \{3;4;5\}$$

$$\nu_i = \left|\left\{ j : \xi_j \in C_i \right\}\right|, \mu_i = \left|\left\{ j : \eta_j \in C_i \right\}\right|, \; i = 1,2,$$

$$\nu_1 = 58, \nu_2 = 28, \mu_1 = 5, \mu_2 = 6, n = 86, m = 11$$

A tesztstatisztika:

$$\chi^2 = 86 \cdot 11 \left(\frac{\left(\frac{58}{86} - \frac{5}{11}\right)^2}{58+5} + \frac{\left(\frac{28}{86} - \frac{6}{11}\right)^2}{28+6} \right) = 2.071$$

$$P(\chi_1^2 > 2.71) = 10\% \Rightarrow$$

Nem tudjuk elutasítani az egyforma képesség hipotézisét!

χ-négyzet próba függetlenségvizsgálatra

- H_0 hipotézis: az A_1, A_2 , ..., A_r és B_1, B_2 , ..., B_s teljes eseményrendszerekre teljesül a függetlenség.

$$\sum_{i,j} \frac{\left(\nu_{ij} - np_i q_j\right)^2}{np_i q_j}$$

- Kritikus tartomány: ha a statisztika értéke nagyobb, mint az rs-1 szabadságfokú χ-négyzet eloszlás 1- α kvantilise, elutasítjuk a nullhipotézist.

Becsléses eset

- Általában, ha az illesztendő eloszlást nem ismerjük – csak a családját - becsüljük a paramétereit. Ekkor a próbastatisztika szabadságfoka annyival csökken, ahány paramétert becsültünk.
- Függetlenségvizsgálatnál általában nem ismerjük a teljes eseményrendszer tagjainak valószínűségét, így r-1+s-1 valószínűséget kell becsülnünk. A szabadságfok ekkor tehát

  rs-1-r-s+2=(r-1)(s-1).

ν_{ij} : $A_i B_j$ gyakorisága

$\nu_{i\bullet}$: A_i gyakorisága

$\nu_{\bullet j}$: B_j gyakorisága

A tesztstatisztika

$$n \sum_{i,j} \frac{\left(\nu_{ij} - \frac{\nu_{i\bullet} \nu_{\bullet j}}{n} \right)^2}{\nu_{i\bullet} \nu_{\bullet j}} \xrightarrow[n \to \infty]{} \chi^2_{(r-1)(s-1)}$$

$r = s = 1$ esetben

$$n \frac{\left(\nu_{11} \nu_{22} - \nu_{12} \nu_{21} \right)^2}{\nu_{1\bullet} \nu_{2\bullet} \nu_{\bullet 1} \nu_{\bullet 2}} \xrightarrow[n \to \infty]{} \chi^2_1$$

Mediterranean Diet and Health

Research conducted by: De Longerill et al.

Case study prepared by: David Lane and Emily Zitek

Overview
Most doctors would probably agree that a Mediterranean diet, rich in vegetables, fruits, and grains, is healthier than a high-saturated fat diet. Indeed, previous research has found that the diet can lower risk of heart disease. However, there is still considerable uncertainty about whether the Mediterranean diet is superior to a low-fat diet recommended by the American Heart Association. This study is the first to compare these two diets.

The subjects, 605 survivors of a heart attack, were randomly assigned follow either (1) a diet close to the "prudent diet step 1" of the American Heart Association (control group) or (2) a Mediterranean-type diet consisting of more bread and cereals, more fresh fruit and vegetables, more grains, more fish, fewer delicatessen foods, less meat. An experimental canola-oil-based margarine was used instead of butter or cream. The oils recommended for salad and food preparation were canola and olive oils exclusively. Moderate red wine consumption was allowed.

Over a four-year period, patients in the experimental condition were initially seen by the dietician, two months later, and then once a year. Compliance with the dietary intervention was checked by a dietary survey and analyses of plasma fatty acids. Patients in the control group were expected to follow the dietary advice given by their physician.

The researchers collected information on number of deaths from cardiovascular causes e.g., heart attack, strokes, as well as number of nonfatal heart-related episodes. The occurrence of malignant and nonmalignant tumors was also carefully monitored.

Questions to Answer
Is the Mediterranean diet superior to a low-fat diet recommended by the American Heart Association?

Design Issues

The strength of the design is that subjects were randomly assigned to conditions. A possible weakness is that compliance rates depended on reports rather than observation since observation is impractical in this type of reseach.

Descriptions of Variables

Variable	Description
Type of diet	AHA or Mediterranean
Various outcome measures of health and disease	does the patient have cancer, etc.?

References

De Longerill, M., Salen, P., Martin, J., Monjaud, I., Boucher, P., Mamelle, N. (1998). Mediterranean Dietary pattern in a Randomized Trial. *Archives of Internal Medicine, 158*, 1181-1187.

Links

Another study on the Mediterrainean Diet

Exercises

1. What percentage of people on the AHA diet had some sort of illness or death?
2. What percentage of people on the Mediterranean diet had some sort of illness or death?
3. Conduct a Pearson Chi-Square test to determine if there is any relationship between diet and outcome.
4. Compute a 95% confidence interval on the proportion of people who are healthy on the AHA diet.

Frequencies

	Cancers	Deaths	Nonfatal illness	Healthy	Total
AHA	15	24	25	239	303
Mediterranean	7	14	8	273	302
Total	22	38	33	512	605

Show Data | Open Data with Excel | Analysis Lab

Contingency Tables

Prerequisites
Chi Square Distribution, One-Way Tables

Learning Objectives
1. State the null hypothesis tested concerning contingency tables
2. Compute expected cell frequencies
3. Compute Chi Square and df

This section shows how to use Chi Square to test the relationship between nominal variables for significance. For example, Table 1 shows the data fro the Mediterranean Diet and Health case study.

Table 1. Frequencies for Diet and Health Study

	Outcome				
Diet	Cancers	Fatal Heart Disease	Non-Fatal Heart Disease	Healthy	Total
AHA	15	24	25	239	303
Mediterranean	7	14	8	273	302
Total	22	38	33	512	605

The question is whether there is a significant relationship between diet and outcome. The first step is to compute the expected frequency for each cell based on the assumption that there is no relationship. These expected frequencies are computed from the totals as follows. We begin by computin the expected frequency for the AHA Diet/Cancers combination. Note that 22/605 subjects developed cancer. The proportion who developed cancer is therefore 0.0364. If there were no relationship between diet and outcome, then we would expect 0.0364 of those on the AHA diet to develop cancer. Since 303 subjects were on the AHA diet, we would expect (0.0364)(303) = 11.02 cancers on the AHA diet. Similarly, we would expect (0.0364)(302) = 10.98 cancers on the Mediterranean diet. In general, the expected frequenc for a cell in the ith row and the jth column is equal to

$$E_{i,j} = \frac{T_i T_j}{T}$$

where $E_{i,j}$ is the expected frequency for cell i,j, T_i is the total ith row, T_j is

the total for the jth column, and T is the total number of observations. For the AHA Diet/Cancers cell, i = 1, j = 1, T_i = 303, T_j = 22, and T = 605. Table 2 shows the expected frequencies (in parenthesis) for each cell in the experiment. Table 2 shows the expected frequencies (in parenthesis) for each cell in the experiment.

Table 2. Observed and Expected Frequencies for Diet and Health Study

	Outcome				
Diet	Cancers	Fatal Heart Disease	Non-Fatal Heart Disease	Healthy	Total
AHA	15 (11.02)	24 (19.03)	25 (16.53)	239 (256.42)	303
Mediterranean	7 (10.98)	14 (18.97)	8 (16.47)	273 (255.58)	302
Total	22	38	33	512	605

The significance test is conducted by computing Chi Square as follows.

$$\chi^2_3 = \sum \frac{(E-O)^2}{E} = 16.55$$

The degrees of freedom is equal to (r-1)(c-1) where r is the number of rows and c is the number of columns. For this example, the degrees of freedom is (2-1)(4-1) = 3. The Chi Square calculator can be used to determine that the probability value for a Chi Square of 16.55 with three degrees of freedom is less 0.0009. Therefore, the null hypothesis of no relationship between diet and outcome can be rejected.

A key assumption of the Chi Square test of independence is that each subject contributes data to only one cell. Therefore the sum of all cell frequencies in the table must be the same as the number of subjects in the experiment. Consider an experiment in which each of 16 subjects each attempted two anagram problems. The data are shown in Table 3.

Table 3. Anagram Problem Data

	Anagram 1	Anagram 2

Solved	10	4
Did not Solve	6	12

It would not be valid to use the Chi Square test on these data since each subject contributed data to two cells: one cell based on their performance c Anagram 1 and one cell based on their performance on Anagram 2. The tot of the cell frequencies in the table is 32 but the total number of subjects is only 16.

The formula for Chi Square yields a statistic that is only approximately Chi Square distribution. In order for the approximation to be adequate, the total number of subjects should be at least 20. Some authors claim that the correction for continuity should be used whenever an expected cell frequency is below 5. Research in statistics has shown that this practice is not advisable. For example, see:

Bradley, D. R., Bradley, T. D., McGrath, S. G., & Cutcomb, S. D. (1979) Type I error rate of the chi square test of independence in r x c tables that have small expected frequencies. *Psychological Bulletin*, *86*, 1200-1297.

The correction for continuity when applied to 2 x 2 contingency tables is called the Yates correction. The simulation 2 x 2 tables lets you explore the accuracy of the approximation and the value of this correction.